28. Narrowband Noise Representation

In most communication systems, we are often dealing with band-pass filtering of signals. Wideband noise will be shaped into bandlimited noise. If the bandwidth of the bandlimited noise is relatively small compared to the carrier frequency, we refer to this as *narrowband noise*. Figure 28.1 shows how to generate narrowband noise from wideband noise.

Figure 28.1 Generation of narrowband noise.

We can derive the power spectral density $G_n(f)$ and the *auto-correlation function* $R_{nn}(\tau)$ of the narrowband noise and use them to analyse the performance of linear systems. In practice, we often deal with mixing (multiplication), which is a non-linear operation, and the system analysis becomes difficult. In such a case, it is useful to express the narrowband noise as

$$n(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \qquad (28.1)$$

where f_c is the carrier frequency within the band occupied by the noise. $x(t)$ and $y(t)$ are known as the *quadrature components* of the noise $n(t)$. The Hibert transform of $n(t)$ is

$$\hat{n}(t) = \mathcal{H}[n(t)] = x(t) \sin 2\pi f_c t + y(t) \cos 2\pi f_c t \qquad (28.2)$$

Proof.

The Fourier transform of $n(t)$ is

$$N(f) = \frac{1}{2} X(f - f_c) + \frac{1}{2} X(f + f_c) + \frac{1}{2} jY(f - f_c) - \frac{1}{2} jY(f + f_c)$$

Let $\hat{N}(f)$ be the Fourier transform of $\hat{n}(t)$. In the frequency domain, $\hat{N}(f) = N(f)[-j \operatorname{sgn}(f)]$. We simply multiply all positive frequency components of $N(f)$ by $-j$ and all negative frequency components of $N(f)$ by j. Thus,

$$\hat{N}(f) = -j\frac{1}{2} X(f-f_c) + j\frac{1}{2} X(f+f_c) - j\frac{1}{2} jY(f-f_c) - j\frac{1}{2} jY(f+f_c)$$
$$\hat{N}(f) = -j\frac{1}{2} X(f-f_c) + j\frac{1}{2} X(f+f_c) + \frac{1}{2} Y(f-f_c) + \frac{1}{2} Y(f+f_c)$$

and the inverse Fourier transform of $\hat{N}(f)$ is

$$\hat{n}(t) = x(t) \sin 2\pi f_c t + y(t) \cos 2\pi f_c t$$ Q.E.D.

The quadrature components $x(t)$ and $y(t)$ can now be derived from equations (28.1) and (28.2).

$$x(t) = n(t)\cos 2\pi f_c t + \hat{n}(t)\sin 2\pi f_c t \quad (28.3)$$

and

$$y(t) = n(t)\cos 2\pi f_c t - \hat{n}(t)\sin 2\pi f_c t \quad (28.4)$$

Given $n(t)$, the quadrature components $x(t)$ and $y(t)$ can be obtained by using the arrangement shown in Figure 28.2.

Figure 28.2 Generation of quadrature components of $n(t)$.

$x(t)$ and $y(t)$ have the following properties:

1. $E[x(t)\,y(t)] = 0$. $x(t)$ and $y(t)$ are uncorrelated with each other.
2. $x(t)$ and $y(t)$ have the same means and variances as $n(t)$.
3. If $n(t)$ is Gaussian, then $x(t)$ and $y(t)$ are also Gaussian.
4. $x(t)$ and $y(t)$ have identical power spectral densities, related to the power spectral density of $n(t)$ by

$$G_x(f) = G_y(f) = G_n(f - f_c) + G_n(f + f_c) \quad (28.5)$$

for $f_c - 0.5B < |f| < f_c + 0.5B$ and B is the bandwidth of $n(t)$.

Proof. (Under Construction)

Equation (28.5) is the key that will enable us to calculate the effect of noise on AM and FM systems. It implies that the power spectral density of $x(t)$ and $y(t)$ can be found by shifting the positive portion and negative portion of $G_n(f)$ to zero frequency and adding to give $G_x(f)$ and $G_y(f)$. This is shown in Figure 28.3.

Figure 28.3 (a) Power spectral density of $n(t)$, (b) Power spectral density of $x(t)$ and $y(t)$.

In the special case where $G_n(f)$ is symmetrical about the carrier frequency f_c, the positive- and negative-frequency contributions are shifted to zero frequency and added to give

$$G_x(f) = G_y(f) = 2G_n(f - f_c) = 2G_n(f + f_c) \tag{28.6}$$

Example 28.1

Given that the power spectral density of a narrowband Gaussian noise of variance σ^2 and power N is

$$G_n(f) = \frac{N}{2}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(f-f_c)^2}{2\sigma^2}} + \frac{N}{2}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(f+f_c)^2}{2\sigma^2}}$$

where f_c is the carrier frequency within the band occupied by the noise, then the power spectral densities of the quadrature components of the noise are

$$G_x(f) = G_y(f) = 2G_n(f + f_c) = N\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{f^2}{2\sigma^2}}$$

This is shown in Figure 28.4.

Figure 28.4 (a) Power spectral density of narrowband Gaussian noise $n(t)$, (b) Power spectral density of $x(t)$ and $y(t)$.

Performance of Binary FSK

Figure 28.5 Synchronous detection of binary FSK signals.

Consider the synchronous detector of binary FSK signals shown in Figure 28.5. In the presence of *additive white Gaussian noise* (AWGN), $w(t)$, the received signal is

$$r(t) = A\cos 2\pi f_{c_1} t + w(t)$$

where A is a constant and f_{c_1} is the carrier frequency employed if a 1 has been sent. The signals at the output of the band-pass filters of centre frequencies f_{c_1} and f_{c_2} are

$$r_1(t) = A\cos 2\pi f_{c_1} t + n_1(t)$$

and

$$r_2(t) = n_2(t)$$

where

$$n_1(t) = x_1(t)\cos 2\pi f_{c_1}t - y_1(t)\sin 2\pi f_{c_1}t$$

and

$$n_2(t) = x_2(t)\cos 2\pi f_{c_2}t - y_2(t)\sin 2\pi f_{c_2}t$$

are the narrowband noise. With appropriate design of low-pass filter and sampling period, the sampled output signals are

$$v_{o1} = A + x_1$$

$$v_{o2} = x_2$$

and

$$v = A + [x_1 - x_2].$$

x_1 and x_2 are statistically independent Gaussian random variables with *zero mean* and fixed *variance* $\sigma^2 = N$, where N is the power of the random variable. It can be seen that one of the detectors has signal plus noise, the other detector has noise only.

When f_{c_2} is the carrier frequency employed for sending a 0, the received signal is

$$r(t) = A\cos 2\pi f_{c_2}t + w(t).$$

It can be shown that

$$v = -A + [x_1 - x_2]$$

Since $E[x_1 - x_2]^2 = E[x_1]^2 - 2E[x_1x_2]^2 + E[x_2]^2 = E[x_1]^2 + E[x_2]^2 = \sigma^2 + \sigma^2$, the total variance $\sigma_t^2 = 2\sigma^2$. The two distributions of v are shown in Figure 28.6.

Figure 28.6 Conditional probability density function.

The conditional probability density function of v assuming a 0 is sent is

$$f(v/0) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{(v+A)^2}{2\sigma_t^2}}$$

and the probability of error given a 0 is sent is

$$\begin{aligned} P_{e0} &= \int_0^\infty f(v/0)dv \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{(v+A)^2}{2\sigma_t^2}} dv \end{aligned} \qquad (28.7)$$

Similarly, the probability of error given a 1 is sent is

$$\begin{aligned} P_{e1} &= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{(v-A)^2}{2\sigma_t^2}} dv \\ &= P_{e0} \end{aligned}$$

Let p_0 be the probability of sending a 0 and p_1 be the probability of sending a 1. For equally likely transmission of binary signals, we have $p_0 = p_1 = 0.5$. The average probability of error is given by

$$\begin{aligned} P_e &= p_0\, P_{e0} + p_1\, P_{e1} \\ &= P_{e0} \end{aligned}$$

Let $u = \frac{v+A}{\sqrt{2}\sigma_t}$. Then $u^2 = \frac{(v+A)^2}{2\sigma_t^2}$ and $du = \frac{dv}{\sqrt{2}\sigma_t}$. Substituting u and du into equation (28.7), we get

$$\begin{aligned} P_e &= \int_{u=\frac{A}{\sqrt{2}\sigma_t}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2}\, du \\ &= \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_u^\infty e^{-u^2}\, du \right] \end{aligned} \qquad (28.8)$$

Equation (28.8) becomes

$$
\begin{aligned}
P_e &= \frac{1}{2}\ erfc\left(\frac{A}{\sqrt{2}\sigma_t}\right) \\
&= \frac{1}{2}\ erfc\left(\frac{A}{2\sigma}\right) \\
&= \frac{1}{2}\ erfc\left(\frac{A}{2\sqrt{N}}\right).
\end{aligned}
$$

Similarly, we can use this approach to derive the average probability of error for BASK and BPSK systems. In the BASK system, the synchronous detector output consists of a signal A plus noise or a noise alone. In the BPSK system, the synchronous detector output consists of a polar signal $\pm A$ plus noise. The results are summarised in Table 28.1

BPSK	$P_e = \frac{1}{2}\ erfc\left(\frac{A}{\sqrt{2N}}\right)$
BFSK	$P_e = \frac{1}{2}\ erfc\left(\frac{A}{2\sqrt{N}}\right)$
BASK	$P_e = \frac{1}{2}\ erfc\left(\frac{A}{2\sqrt{2N}}\right)$

Table 28.1 Performance of various modulation systems.

References

[1] J. G. Porakis and M. Salehi, Communication Systems Engineering, Prentice Hall, 1994. ISBN 0-13-158932-6.

[2] M. S. Roden, Analog and Digital Communication Systems, 3/e, Prentice Hall, 1991, ISBN 0-13-033325-5.

[3] H. Taub and D. L. Schilling, Principles of Communication Systems, 2/e, MaGraw-Hill, 1996, ISBN 0-07-062955-2.

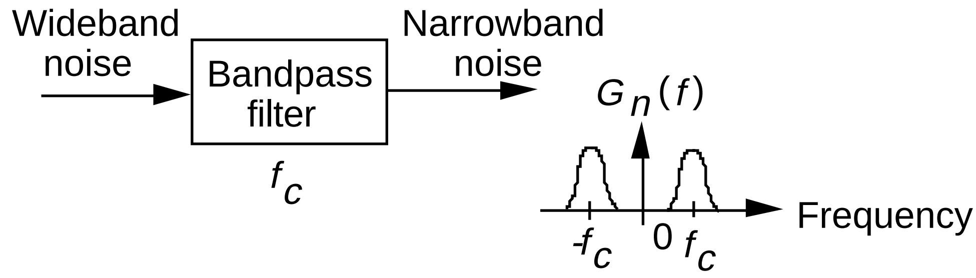

Figure 28.1 Generation of narrowband noise.

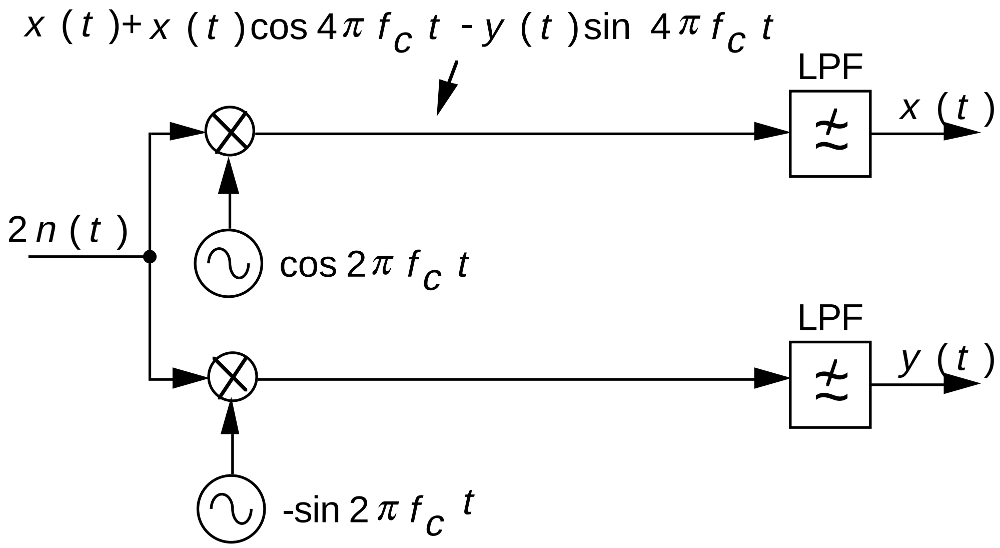

Figure 28.2 Generation of quadrature components of $n(t)$.

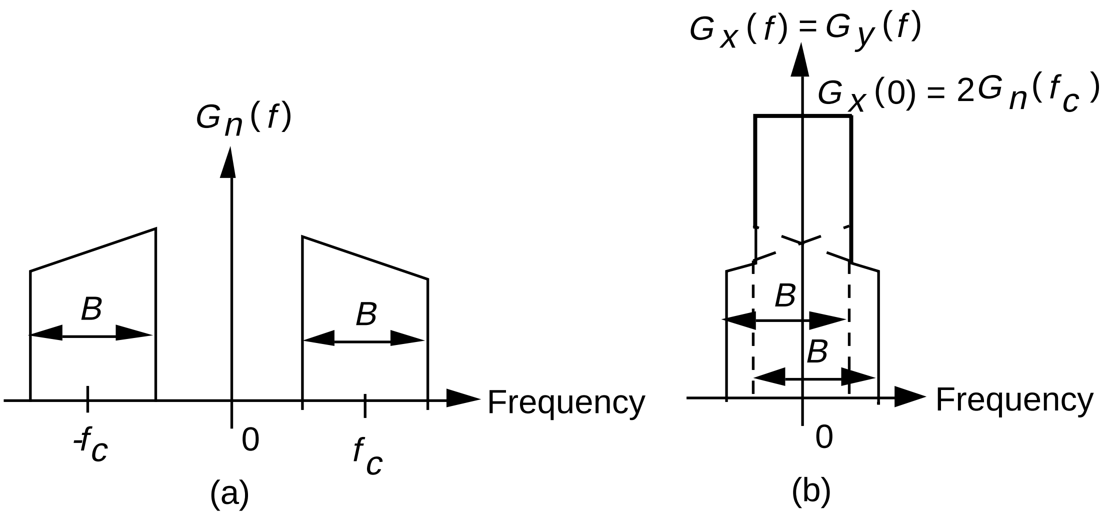

Figure 28.3 (a) Power spectral density of $n(t)$. (b) Power spectral density of $x(t)$ and $y(t)$.

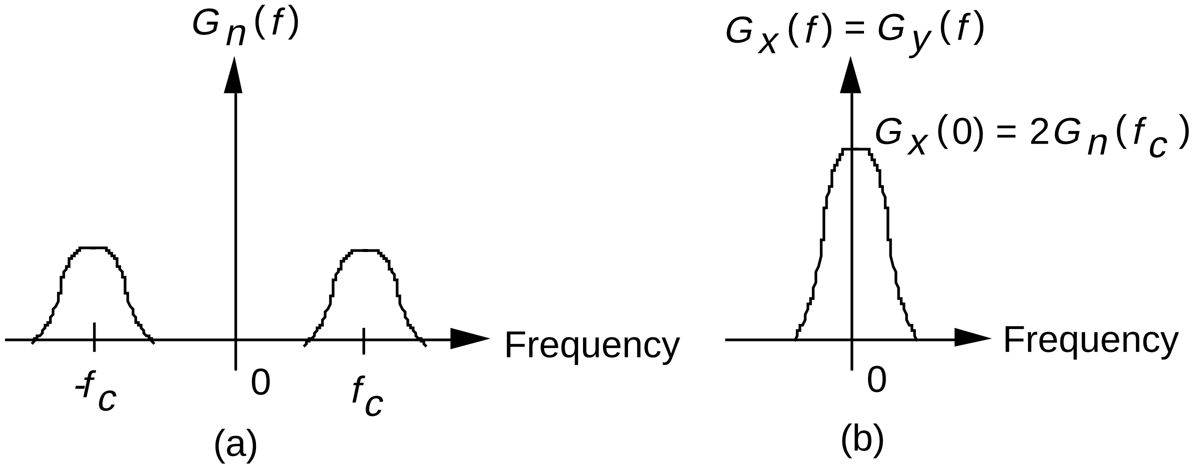

Figure 28.4 (a) Power spectral density of narrowband Gaussian noise $n(t)$. (b) Power spectral density of $x(t)$ and $y(t)$.

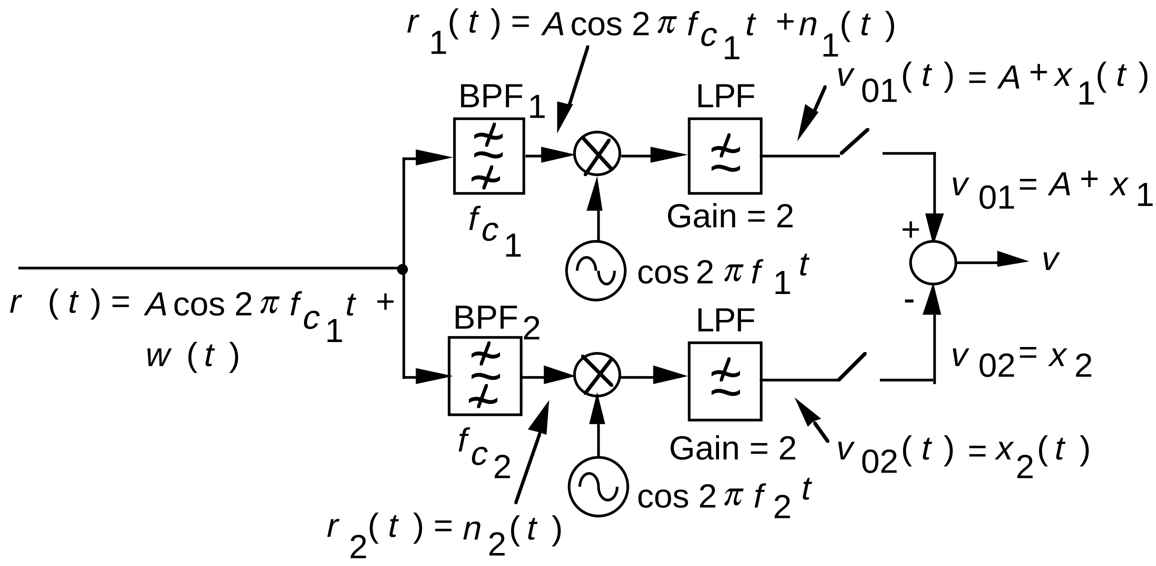

Figure 28.5 Synchronous detection of binary FSK signals.

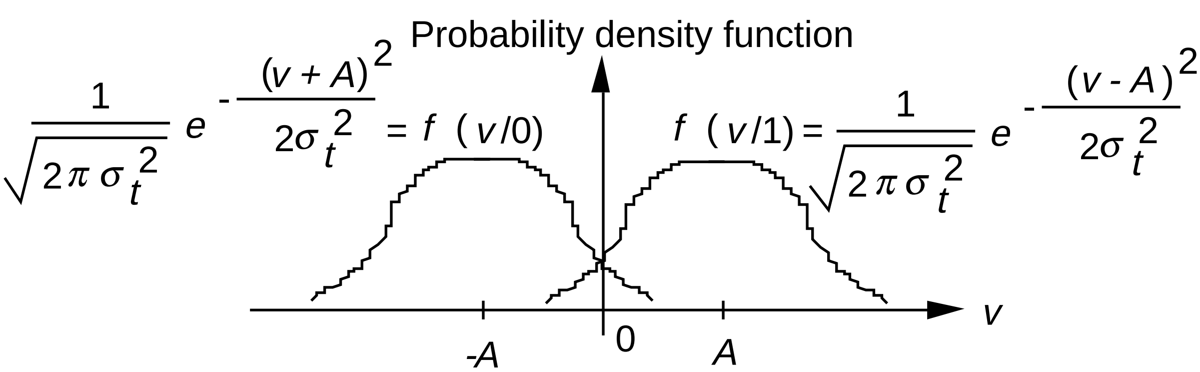

Figure 28.6 Conditional probability density function.